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一类新的2+1维非线性发展方程及其解的对合表示

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摘要: 由线性谱问题的相容性条件得到一个新的2+1维非线性发展方程。利用位势函数与特征函数之间的约束获得Bargmann系统,通过Euler-Lagrange方程及Legendre变换构造Jacobi-Ostrogradsky坐标。应用Lax对非线性化方法,生成了一个新的有限维Hamilton正则系统。最后证明其为Liouville意义下完全可积系统,并得到发展方程族的对合表示。

关键词: 谱问题; Lax对非线性化; Bargmann系统; 可积系统; 对合表示

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0 引言

无穷维发展方程的可积性是当今数学物理研究的一个重要方面。将无穷维系统转化为有限维完全可积系统是证明可积性的基本思想之一[1]。Lax对非线性化方法是获得有限维可积系统的有效工具。近年来非线性化方法被用于求解高维孤子方程[2],如2+1维Toda方程、2+1维K-P方程等。现研究一个新的2+1维非线性发展方程。首先通过两个二阶谱问题之间的相容性条件获得非线性发展方程,得到其Lax对。利用Bargmann约束将谱问题转化为一个Bargmann系统,并将Lax对非线性化,获得一个有限维Hamilton正则系统,并证明此系统在Liouville意义下是完全可积的。

1 二阶谱问题及非线性发展方程的Lax对

讨论二阶谱问题

$$-\partial^2\varphi+\lambda u\varphi+\lambda\nu\partial\varphi=0 \tag{1}$$

式中,$\partial=\partial_x$ 且 $\partial^{-1}\partial=1$;$u=u(x,t)$;$\nu=\nu(x,t)\in R$ 为该谱问题的位势函数;λ 为特征值;φ 为对应于 λ 的特征函数。

在基础空间 $\Omega=(-\infty,+\infty)$ 上讨论特征值问题(1),假设 u,ν,φ 为无穷远速降的函数。

命题1[3]:

(1) 如下二阶谱问题。

$$\begin{cases}-\partial^2\varphi+\lambda u\varphi+\lambda\nu\partial\varphi=0\\ -\partial^2\psi+\lambda u\psi+\lambda\partial(\nu\psi)=0\end{cases} \tag{2}$$

构成完整谱系。

(2) 若 φ,ψ 是谱系(2)的特征值 λ 所对应的特征函数,则

$$\mathrm{grad}\lambda=\begin{pmatrix}\delta\lambda/\delta u\\ \delta\lambda/\delta\nu\end{pmatrix}=-\left(\int_\Omega(u\varphi\psi+\nu\varphi_x\psi)\,\mathrm{d}x\right)^{-1}\begin{pmatrix}\lambda\varphi\psi\\ \lambda\varphi_x\psi\end{pmatrix}。$$

设辅谱问题为 $\varphi_t=W\varphi$,其中,$W=\sum_{j=0}^{+\infty}(a_j+b_j\partial)\lambda^{-j}$。

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令 $g_j = (-b_j, a_j)^{\mathrm{T}}, j = 0, 1, 2, \cdots$，则由相容性条件 $\varphi_{xxt} = \varphi_{txx}$ 得如下递推关系

$$Kg_j = Jg_{j+1}, j = 0, 1, 2, \cdots,$$

其中

$$K = \begin{pmatrix} 0 & \partial^2 \\ -\partial^2 & 2\partial \end{pmatrix}, J = \begin{pmatrix} u\partial + \partial u & \nu\partial \\ \partial\nu & 0 \end{pmatrix}。$$

直接计算可知

$$K \operatorname{grad}\lambda = \lambda J \operatorname{grad}\lambda \tag{3}$$

取 $g_0 = \left(-\frac{1}{\nu}, \frac{u}{\nu^2}\right)^{\mathrm{T}}$ 则有 $Jg_0 = 0$，并由 (3) 得到递推序列 $\{g_j, j = 1, 2, \cdots\}$，特别 $g_1 = \left(\frac{2u - \nu_x}{\nu^3}, \frac{u_x}{\nu^3} - \frac{3u^2}{\nu^4}\right)^{\mathrm{T}}$。

定理 1: 在等谱条件($\lambda_t = 0$) 下，非线性发展方程族为

$$\begin{pmatrix} u_t \\ \nu_t \end{pmatrix} = Kg_m = Jg_{m+1} \quad m = 0, 1, 2, \cdots, \tag{4}$$

对应的 Lax 对为

$$\begin{cases} L\varphi = \lambda\varphi \\ \varphi_{t_m} = W_m\varphi, \ m = 0, 1, 2, \cdots \end{cases} \tag{5}$$

式中，$W_m = (\lambda^{m+1}W)_+ = \sum_{j=0}^{m}(a_j + b_j\partial)\lambda^{m-j+1}$。

特别，当 $m = 0$ 时，发展方程为

$$\begin{cases} u_{t_0} = \left(\frac{u}{\nu^2}\right)_{xx} \\ \nu_{t_0}\left(\frac{1}{\nu}\right)_{xx} + 2\left(\frac{u}{\nu^2}\right)_{xx} \end{cases} \tag{6}$$

当 $m = 1$ 时，发展方程为

$$\begin{cases} u_{t_1} = \left(\frac{u_{xx}}{\nu^3} - 3\frac{u_x\nu_x}{\nu^4} - \frac{6uu_x}{\nu^4} + 12\frac{u^2\nu_x}{\nu^5}\right)_x \\ \nu_{t_1} = \left(\frac{\nu_{xx}}{\nu^3} + \frac{6u\nu_x - 6u^2 - 3\nu_x^2}{\nu^4}\right)_x \end{cases} \tag{7}$$

令 $t_0 = y, t_1 = t$，则得到一个新的 2 + 1 维发展方程

$$\begin{cases} u_t = \left(\frac{u_{xx}}{\nu^3} - 3\frac{u_x\nu_x}{\nu^4} - \frac{6u}{\nu^2}\partial^{-1}u_y\right)_x \\ \nu_t = \left(\frac{\nu_{xx}}{\nu^3} - 3\frac{u\nu_x}{\nu^4} - 3\frac{u - \nu_x}{\nu^2}\partial^{-1}\nu_y\right)_x \end{cases} \tag{8}$$

2　Bargmann 约束与 Hamilton 正则系统

设谱系(2) 的 N 个不同的特征值为 $\lambda_1 < \lambda_2 < \cdots < \lambda_N, \varphi_j, \psi_j$ 为对应于 $\lambda_j (j = 1, 2, \cdots, N)$ 的特征函数[4]，令 $\Lambda = \operatorname{diag}(\lambda_1, \lambda_2, \cdots, \lambda_N), \varphi = (\varphi_1, \varphi_2, \cdots, \varphi_N)^{\mathrm{T}}, \psi = (\psi_1, \psi_2, \cdots, \psi_N)^{\mathrm{T}}$。

命题 2: 设 $G_j = (<\Lambda^{j+1}\varphi, \psi>, <\Lambda^{j+1}\varphi_x, \psi>)$，则 $KG_j = JG_{j+1}$。

证明: 由式(4) 直接计算可得。

令 $g_0 = G_0$，得到 Bargmann 约束

$$u = <\Lambda\varphi, \psi>^{-2} <\Lambda\varphi_x, \psi>, \nu = -<\Lambda\varphi, \psi>^{-1} \tag{9}$$

由此，谱系(2) 等价于如下 Bargmann 系统

$$\begin{cases} -\varphi_{xx} + <\Lambda\varphi,\psi>^{-2}<\Lambda\varphi_x,\psi>\Lambda\varphi - <\Lambda\varphi,\psi>^{-1}\Lambda\ \varphi_x = 0 \\ -\psi_{xx} + <\Lambda\varphi,\psi>^{-2}<\Lambda\varphi_x,\psi>\Lambda\psi + \Lambda(<\Lambda\varphi,\psi>^{-1}\psi)_x = 0 \end{cases} \tag{10}$$

现在寻求与 Bargmann 系统(10)对应的 Hamilton 系统的 J-O 坐标。

定义 Lagrange 函数:$\hat{I} = \int_\Omega I\mathrm{d}x$,其中,$I = <\varphi_x,\psi_x> - <\Lambda\varphi,\psi>^{-1}<\Lambda\varphi_x,\psi>$。

命题3:Bargmann 系统(10)等价于如下 Euler-Lagrange 方程:$\frac{\delta\hat{I}}{\delta\varphi} = 0$,$\frac{\delta\hat{I}}{\delta\psi} = 0$。

辛空间($\omega = \sum_{j=1}^{2}\mathrm{d}q_j \wedge \mathrm{d}p_j, R^{4N}$)上的 Poisson 括号定义为:[5] $\{F,H\} = \sum_{j=1}^{2}\sum_{k=1}^{N}\left(\frac{\partial F}{\partial q_{jk}}\frac{\partial H}{\partial p_{jk}} - \frac{\partial F}{\partial q_{jk}}\frac{\partial H}{\partial p_{jk}}\right)$。

设 q_1,q_2 为广义坐标,p_1,p_2 为广义动量,Hamiltom 函数 $H = \sum_{j=1}^{2} <q_{jx},p_j> - I$,则 q_1,q_2,p_1,p_2 满足如下方程[5]

$$\begin{cases} q_{jx} = \{q_j,H\} = \frac{\partial H}{\partial p_j} \\ p_{jx} = \{p_j,H\} = -\frac{\partial H}{\partial q_j} \end{cases} \quad j = 1,2。$$

由 $H = \sum_{j=1}^{2} <q_{jx},p_j> - I$,知 $\mathrm{d}H = \sum_{j=1}^{2}(<q_{jx},\mathrm{d}p_j> + <\mathrm{d}q_{jx},p_j>) - \mathrm{d}I$,

另一方面,由微分形式不变性知,$\mathrm{d}H = \sum_{j=1}^{2}\left(<\frac{\partial H}{\partial p_j},\mathrm{d}q_j> + <\frac{\partial H}{\partial q_j},\partial p_j>\right) = \sum_{j=1}^{2}(-<p_{jx},\mathrm{d}q_j> + <q_{jx},\mathrm{d}q_j>)$,所以,$\sum_{j=1}^{2}(<p_{jx},\mathrm{d}q_j> + <p_j,\mathrm{d}q_{jx}>) = \mathrm{d}I$。令 $q_1 = \varphi$,$q_2 = \psi$,直接计算得 $p_1 = \psi_x - <\Lambda\varphi,\psi> - 1\Lambda\psi$,$p_2 = \varphi_x$。所以,$H = <p_2,p_1> + <\Lambda q_1,q_2>^{-1}<\Lambda q_2,p_2>$。

构造 Jacobi-Ostrogradsky 坐标如下

$$\begin{cases} y_1 = q_1, \quad y_2 = p_2, \\ z_1 = p_1, \quad z_2 = -q_2。 \end{cases} \tag{11}$$

在坐标系(11)下,Bargmann 约束化为

$$u = -<\Lambda y_1,z_2>^{-2}<\Lambda y_2,z_2>,\ v = <\Lambda y_1,z_2>^{-1} \tag{12}$$

Bargmann 系统(10)等价于如下 Hamilton 系统

$$\begin{cases} y_{jx} = \{y_j,H\} = \frac{\partial H}{\partial z_j}, \\ z_{jx} = \{z_j,H\} = \frac{\partial H}{\partial y_j}。 \end{cases} \quad j = 1,2。 \tag{13}$$

此时

$$H = <y_2,z_1> + <\Lambda y_1,z_2>^{-1}<\Lambda y_2,z_2> \tag{14}$$

3 Hamilton 系统的完全可积性

定理2:在 Jacobi-Ostrogradsky 坐标(11)下,发展方程族(4)所对应的 Lax 对(5)等价于如下形式[3,6,7]

$$\begin{cases} \boldsymbol{Y}_x = \boldsymbol{MY}, \boldsymbol{Z}_x = -\boldsymbol{M}^{\mathrm{T}}\boldsymbol{Z}; \\ \boldsymbol{Y}_{t_m} = \boldsymbol{W}_m\boldsymbol{Y}, \boldsymbol{Z}_{t_m} = -\boldsymbol{W}_m^{\mathrm{T}}\boldsymbol{Z}; \quad m = 0,1,\cdots \end{cases} \tag{15}$$

式中,$\boldsymbol{Y} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} \varphi \\ \varphi_x \end{pmatrix}$;$\boldsymbol{Z} = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} \psi_x - <\Lambda\varphi,\psi>^{-1}\Lambda\psi \\ -\psi \end{pmatrix}$;$\boldsymbol{M} = \begin{pmatrix} 0 & E \\ \Lambda_u & \Lambda_v \end{pmatrix}_{2N\times 2N}$;$\boldsymbol{E} = \boldsymbol{E}_{N\times N} = \mathrm{diag}(1,1,\cdots,1)$;$\boldsymbol{W}_m = (\omega_{ij}^m)_{2N\times 2N}$,$i,j = 1,2$, $m = 0,1,2,\cdots$;$\omega_{11}^m = \sum_{j=0}^{m} a_j\Lambda^{m-j+1}$;$\omega_{12}^m = \sum_{j=0}^{m} b_j\Lambda^{m-j+1}$;$\omega_{21}^m = \sum_{j=0}^{m}(a_{jx} +$

$ub_j\Lambda)\Lambda^{m-j+1}$; $\omega_{22}^m = \sum_{j=0}^m(a_j + b_{jx} + \nu b_j\Lambda)\Lambda^{m-j+1}$。

由 Bargmann 约束(12),直接计算可得:$\omega_{11}^m = \sum_{j=0}^m - <\Lambda^{j+1}y_2, z_2>\Lambda^{m-j+1}$; $\omega_{12}^m = \sum_{j=0}^m <\Lambda^{j+1}y_1, z_2>\Lambda^{m-j+1}$; $\omega_{21}^m = \sum_{j=0}^m <\Lambda^{j+1}y_2, z_1>\Lambda^{m-j+1} - <\Lambda y_1, z_2>^{-1}<\Lambda y_2, z_2>\Lambda^{m+2} + <\Lambda y_1, z_2>^{-2}<\Lambda y_2, z_2><\Lambda^{m+2}y_1, z_2>\Lambda$; $\omega_{22}^m = \sum_{j=0}^m - <\Lambda^{j+1}y_1, z_1>\Lambda^{m-j+1} - <\Lambda y_1, z_1>^{-1}<\Lambda^{m+2}y_1, z_1>\Lambda + \Lambda^{m+2}$。

定理3:在 Bargmann 约束(12)下,发展方程族(4)的 Lax 对(5)被非线性化为如下 Hamiltom 方程

$$\begin{cases} Y_x = \dfrac{\partial H}{\partial Z}, Z_x = -\dfrac{\partial H}{\partial Y}。\\ Y_{t_m} = \dfrac{\partial H_m}{\partial Z}, Z_{t_m} = -\dfrac{\partial H_m}{\partial Y}。\quad m = 0,1,2,\cdots \end{cases} \tag{16}$$

其中 Hamiltom 函数 H 为式(14),$H_m = -<\Lambda y_1, z_2>^{-1}<\Lambda y_2, z_2><\Lambda^{m+2}y_1, z_2> + <\Lambda^{m+2}y_2, z_2> - \sum_{j=1}^m \begin{vmatrix} <\Lambda^{j+1}y_1, z_1> & <\Lambda^{j+1}y_1, z_2> \\ <\Lambda^{m-j+1}y_2, z_1> & <\Lambda^{m-j+1}y_2, z_2> \end{vmatrix}$。

下面讨论 Hamiltom 系统的完全可积性[7]。

令

$$\begin{cases} E_k^1 = -<\Lambda y_1, z_2>^{-1}<\Lambda y_2, z_2>\lambda_k^2, y_{1k}z_{2k} + \lambda_k^2, y_{2k}z_{2k}; \\ E_k^2 = \sum_{l=1, l\neq k}^N \dfrac{\lambda_k^2}{\lambda_l - \lambda_k}\begin{vmatrix} y_{1k} & y_{1l} \\ y_{2k} & y_{2l}\end{vmatrix}\begin{vmatrix} z_{1k} & z_{1l} \\ z_{2k} & z_{2l}\end{vmatrix}。\end{cases}$$

定理4:

(1)

$$\{E_j^m, E_k^n\} = 0, \forall j,k = 1,2,\cdots,N; m,n = 1,2。\tag{17}$$

(2) $\{dE_j^m, j = 1,2,\cdots,N; m = 1,2\}$ 线性无关;

(3) $\Pi_\mu = \sum_{j=1}^N \dfrac{1}{\mu - \lambda_j}(E_j^1 + E_j^2) = \sum_{m=0}^\infty \mu^{-m-1}H_m, H_m = \sum_{j=1}^N \lambda_j^m(E_j^1 + E_j^2) \quad m = 0,1,2\cdots$。

定理5: Hamiltom 系统(16)在 Liouville 意义下是完全可积系,并且 Hamiltom 相流 $g_{H_m}^{t_m}, g_{H_n}^{t_n}$ 可换。

证明:由(17)式直接计算得 $\{\Pi_\mu, \Pi_\lambda\} = 0, \{H_m, H_n\} = 0, m,n = 0,1,2,\cdots$。

直接计算可知 $\{H, E_j^l\} = 0, l = 1,2; j = 0,1,2,\cdots N$,

所以 $\{H, H_m\} = 0,\quad m = 0,1,2,\cdots$

由 Arnold 定理,Hamiltom 系统(16)是完全可积系,且 Hamiltom 相流可换。

定理6:设 (y_1, y_2, z_1, z_2) 满足 Hamiltom 系统(16),则式(12)

$$u = -<\Lambda y_1, z_2>^{-2}<\Lambda y_2, z_2>, \nu = <\Lambda y_1, z_2>^{-1}$$

为非线性发展方程族(4)的对合解。

特别,如果 $(u(x,y,t), v(x,y,t))$ 是方程(6)和(7)的相容解,则 $(u(x,y,t), v(x,y,t))$ 为2+1维非线性发展方程(8)的解。

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A New (2+1)-dimensional Nonlinear Evolution Equation and the Involutive Representations of the Solutions

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Abstract: A new (2+1)-dimensional nonlinear evolution equation is obtained based on the compatible condition of two linear spectral problems. By means of the constraint condition between the potentials and the eigenvector, the Bargmann system is generated. The Jacobi-Ostrogradsky coordinate system has been found through Euler-Lagrange equation and Legendre transformations. Using the nonlinearization approach of Lax pairs, a new finite-dimensional Hamilton canonical equations are obtained. Finally, it is proved completely integrable systems, and the involutive solutions of the evolution equations are given.

Key words: spectral problem; ninlinearization of Lax pairs; Bargmann system; integrable system; involutive representations

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Analysis and Application of Intelligent Evolution in Printing Production Data

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Abstract: Print production enterprise resource management and the management of the production process based on the ERP system, but the lack of large amounts of data analysis and data mining, this paper adopts the intelligent evolution of thinking, the use of advanced algorithms model of MATLAB powerful data processing capability and MATLAB related to the toolbox of powerful dataanalysis capabilities and data mining tools and analytical models, the evolution of the data and global direct search, and explore its application in the analysis of the macro level and in specific areas of the enterprise data, better ways and means.

Key words: ERP system; data analysis; computational intelligence; global search; genetic algorithm professional toolbox

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